

A GEOMETRIC SCHUR FUNCTOR

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ABSTRACT. We give geometric descriptions of the category $C_k(n, d)$ of polynomial representations over a commutative ring k of GL_n of degree d for $d \leq n$, the Schur functor and Schur-Weyl duality. The descriptions and proofs use a modular version of Springer theory and relationships between the equivariant geometry of the affine Grassmannian and the nilpotent cone for the general linear groups. Motivated by this description, we propose generalizations for an arbitrary connected complex reductive group of the category $C_k(n, d)$ and the Schur functor.

1. INTRODUCTION

1.1. Towards the beginning of the 20th century, Schur defined a functor from polynomial representations of $GL_n(\mathbb{C})$ of degree d to representations of the symmetric group \mathfrak{S}_d , which is an equivalence of categories for $n \geq d$. Later it was observed by Green [Gre80] that Schur's functor is well-defined over any field k , but is not an equivalence if the characteristic of k is less than or equal to d . Green defines the Schur functor as follows. Consider the group GL_n and its polynomial representations. The n -dimensional standard representation E of GL_n is polynomial and homogeneous of degree one. Letting the symmetric group \mathfrak{S}_d act on $E^{\otimes d}$ by permutations defines a functor:

$$\mathrm{Hom}(E^{\otimes d}, -) : C_k(n, d) \rightarrow \mathrm{Mod} - k[\mathfrak{S}_d],$$

where $C_k(n, d)$ denotes the category of polynomial representation of GL_n of degree d .

The categories $C_k(n, d)$ and $\mathrm{Mod} - k[\mathfrak{S}_d]$ are much more complicated when $\mathrm{char} k \leq d$. The resulting categories are not semisimple and even basic facts, for example, the dimensions of the irreducible representations of the symmetric group, are not known in general. The Schur functor has been used as an effective tool to relate structure in the representation theory of the general linear groups with that of the symmetric groups and vice versa.

1.2. In the first part of this paper, we relate the existence of the Schur functor to the geometry of certain singular spaces associated to GL_n . For $n \geq d$, we give a geometric interpretation of the category $C_k(n, d)$ and the Schur functor in terms of Springer theory for the nilpotent cone $\mathcal{N}_d \subset \mathfrak{gl}_d$. More precisely, we show:

Theorem 1.1. *For any $n \geq d$, there is an equivalence of categories:*

$$\phi^\bullet : C_k(n, d) \xrightarrow{\sim} P_{GL_d}(\mathcal{N}_d; k),$$

which takes the representation $E^{\otimes d}$ to the Springer sheaf \mathcal{S} .

Using this we provide a geometric proof of Carter-Lusztig's generalization of Schur-Weyl duality.

Theorem 1.2. [CL74, Thm. 3.1] *For any $d \leq n$, the morphism*

$$k[\mathfrak{S}_d] \rightarrow \mathrm{End}_{G_n^k}(E^{\otimes d})$$

defined by permuting the tensor factors is an isomorphism.

Corollary 1.3. *There is a commutative diagram of functors:*

$$(1.1) \quad \begin{array}{ccc} P_{G_d}(\mathcal{N}) & \xrightarrow{\mathrm{Hom}(\mathcal{S}, -)} & \mathrm{Mod} - \mathrm{End}(\mathcal{S}) \\ \phi^\bullet \downarrow \simeq & & \uparrow \simeq \\ C_k(n, d) & \xrightarrow{\mathrm{Hom}(E^{\otimes d}, -)} & \mathrm{Mod} - k[\mathfrak{S}_d], \end{array}$$

1.3. The proof of these statement is reached by crossing two bridges.

The first bridge is the geometric Satake equivalence, which relates the representation theory of a split reductive group G over k to a category of equivariant perverse sheaves on the affine Grassmannian for the complex reductive group $G_{\mathbb{C}}^{\vee}$ with dual root datum. In particular, it allows us to identify the category of polynomial representations $C_k(n, d)$ with a part of the affine Grassmannian $\overline{\mathcal{G}r^{\mathbf{d}}} \subset \mathcal{G}r_{GL_n}$.

The second bridge is a relationship, which exists for the group GL_n , between the nilpotent cone and affine Grassmannian. In the paper [Lus81], Lusztig introduced a map $\phi : \mathcal{N}_d \rightarrow \overline{\mathcal{G}r^{\mathbf{d}}}$. Using a map in the other direction defined on quotient stacks, we prove an equivalence of categories of equivariant perverse sheaves.

1.4. In the second part of the paper, we shift our focus to an arbitrary connected complex reductive group G . We observe that our reinterpretation of the Schur functor as a functor from the category of adjoint-equivariant perverse sheaves on the nilpotent cone to modules over endomorphisms of the Springer sheaf is well-defined for any reductive group.

Moreover, we give a reformulation in terms of the Fourier-Sato transform \mathbb{T} on the Lie algebra \mathfrak{g} . Let $j_{\mathrm{rs}} : \mathfrak{g}_{\mathrm{rs}} \hookrightarrow \mathfrak{g}$ denote the open embedding of the regular semi-simple locus and $i : \mathcal{N} \hookrightarrow \mathfrak{g}$ the closed embedding of the nilpotent cone. Consider the functor

$$\mathcal{F} := j_{\mathrm{rs}}^* \circ \mathbb{T} \circ i_* : P_G(\mathcal{N}; k) \rightarrow P_G(\mathfrak{g}_{\mathrm{rs}}; k),$$

where $P_G(X; k)$ denotes the category of G -equivariant perverse sheaves. We show that \mathcal{F} factors through $\mathrm{Loc}_W(\mathfrak{g}_{\mathrm{rs}})$, the category of local systems on $\mathfrak{g}_{\mathrm{rs}}$ with monodromy that factors through the Weyl group, and identify \mathcal{F} with the functor $\mathrm{Hom}(\mathcal{S}, -)$.

We propose that $P_G(\mathcal{N}; k)$ should be thought of as a generalization of the category $C_k(n, d)$ and \mathcal{F} as the *geometric Schur functor* for the group G .

1.5. **Related work.** This paper is a revised version of part of the author's Ph.D. thesis [Mau10].

A generalization of some of the results in this paper has since been explored by Achar, Henderson and Riche. In [AH12] and then in [AHR12], they consider a split reductive group G over k and the category of “small” representations of G (analogous to a variant of $C_k(n, d)$). They show that the corresponding part of the affine Grassmannian for the dual group $G_{\mathbb{C}}^{\vee}$ contains an open locus that maps G^{\vee} -equivariantly to the nilpotent cone for G^{\vee} . They use this map to construct a functor Ψ_G between categories of perverse sheaves on the affine Grassmannian and nilpotent cone. The main result of these papers is an equivalence between the

composition of Ψ_G with the geometric Schur functor and the composition of the Satake equivalence with taking the zero weight space together with its W -action.

Another closely-related picture arises in joint work with Achar. Motivated by the equivalence between $C_k(n, d)$ and $P_{GL_n}(\mathcal{N}; k)$, we propose [AM12] a *geometric Ringel functor* for the equivariant derived category $P_G(\mathcal{N}; k)$. The definition is very similar to that of the geometric Schur functor in this paper. In particular, we define a functor

$$\mathcal{R} := i^* \mathbb{T} i_* [\dim \mathcal{N} - \dim \mathfrak{g}] : D_G(\mathcal{N}; k) \rightarrow D_G(\mathcal{N}; k)$$

and show that it is an autoequivalence.

We expect that functors \mathcal{F} and \mathcal{R} will be very useful tools in understanding the categories $P_G(\mathcal{N}; k)$ and $D_G(\mathcal{N}; k)$.

1.6. Here is an outline of the paper. Section 2 contains a summary of the various ingredients that will be used in the paper. In Section 3, we study various maps between the nilpotent cone and affine Grassmannian and the relations they satisfy. We then use these relations in Section 4 to prove an equivalence of categories of perverse sheaves, with which, in Section 5, we deduce Carter-Lusztig's Schur-Weyl duality as a corollary. Sections 6 and 7 contain a proposal for a “geometric Schur functor.”

1.7. **Acknowledgements.** This paper has been a long time coming and so the author had a number of years to benefit from useful conversations and deep insights from many people. He would like to thank in particular: David Ben-Zvi for continued support and advice and Daniel Juteau whose thesis was a source of inspiration for much of this paper. Thanks as well to Pramod Achar, Dennis Gaitsgory, Joel Kamnitzer, David Helm, David Nadler, Catharina Stroppel, Geordie Williamson, Zhiwei Yun, and Xinwen Zhu.

2. DRAMATIS PERSONAE

2.1. **Notation.** Let G be a connected reductive algebraic group over the complex numbers. Let \mathfrak{g} be its Lie algebra, $\mathcal{N} \subset \mathfrak{g}$ the nilpotent cone, \mathfrak{g}_{rs} the regular semi-simple locus and W the Weyl group.

Let $i : \mathcal{N} \hookrightarrow \mathfrak{g}$ denote the closed inclusion of the nilpotent cone in the Lie algebra. Let \mathfrak{g}_{rs} denote the regular semi-simple locus of \mathfrak{g} and $j_{\text{rs}} : \mathfrak{g}_{\text{rs}} \hookrightarrow \mathfrak{g}$ the open inclusion.

Let k be a commutative ring. For a scheme X defined over the complex numbers with an action of a reductive group G , we denote by $P_G(X; k)$ the category of G -equivariant perverse sheaves with coefficients in k on X . This is equivalent to the category of perverse sheaves on the quotient stack $[X/G]$ (cf. [LO09, Rmk. 5.5]).

2.1.1. We denote the general linear group GL_r by G_r and its Lie algebra by \mathfrak{g}_r . We consider the group G_r over various rings and when we need to specify the ring over which we are working, we write G_r^k . We fix the standard upper triangular Borel subgroup B_r of G_r , with its unipotent radical U_r . Let $T_r \cong \mathbb{G}_m^r$ be the Cartan subgroup of diagonal matrices and $\mathfrak{h}_r \cong \mathbb{A}^r$ its Lie algebra, the standard Cartan subalgebra. The Weyl group $W_r = \mathfrak{S}_r$ acts on $\mathfrak{h}_r = \mathbb{A}^r$ by permuting its coordinates. We denote the weight lattice of G_r by Λ_r and identify it with \mathbb{Z}^r using our identification of the Cartan subgroup. The set of highest weights Λ^+ are those $\lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda$ such that $\lambda_i \geq \lambda_j$ for all $i < j$.

We denote the category of homogeneous polynomial representations of G_n of degree d by $C_k(n, d)$. Let $\Lambda(n, d) \subset \Lambda_n$ be the set of weights $(\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $\sum_i \lambda_i = d$ and $\lambda_i \geq 0$ for all $1 \leq i \leq n$. As shown in [Jan03, Prop. A.3], the category of polynomial representations of degree d , $C_k(n, d)$, is precisely the full subcategory of those representations all of whose weights lie in the subset $\Lambda(n, d)$.

2.2. Schur-Weyl Duality and the Schur Functor. The following classical result is known as Schur-Weyl duality.

Theorem 2.1. *Let E denote the standard n -dimensional representation of $G_n^{\mathbb{C}}$. For any $d \leq n$, the morphism $\mathbb{C}[\mathfrak{S}_d] \rightarrow \text{End}_{G_n^{\mathbb{C}}}(E^{\otimes d})$ given by permuting the tensor factors is an isomorphism.*

A generalization of this theorem appears in the work of Carter and Lusztig. They show that this result is true over any commutative ring. Namely,

Theorem 2.2. [CL74, Thm. 3.1] *Consider G_n over an arbitrary commutative ring k . Let E denote the standard n -dimensional representation of G_n^k . For any $d \leq n$, the morphism $k[\mathfrak{S}_d] \rightarrow \text{End}_{G_n^k}(E^{\otimes d})$ given by permuting the tensor factors is an isomorphism.*

We give a new geometric proof of this result in section 5.

Using the symmetric group action on $E^{\otimes d}$, one can define a functor from $C_k(n, d)$ to the category of representations of the symmetric group \mathfrak{S}_d .

Definition 2.3. The Schur functor $\mathcal{S} : C_k(n, d) \rightarrow \text{Rep}_k \mathfrak{S}_d$ is defined as the functor $\text{Hom}(E^{\otimes d}, -)$ on which $k[\mathfrak{S}_d]$ acts on the right.

Remark 2.4. This functor was defined by Schur, who showed that it is an equivalence of categories when k is a field of characteristic greater than d . It is not an equivalence if the characteristic is less than or equal to d .

From this definition, it is clear that \mathcal{S} admits a left adjoint. A slightly different description of the Schur functor also yields a right adjoint (cf. [DEN04]). We interpret the Schur functor geometrically and obtain similar descriptions for its adjoints as well.

Remark 2.5. In what follows, we will give a geometric description of the category $C_k(n, d)$ that does not depend on n . We should point out that it is well-known that for any $n > d$, there is an equivalence $C_k(n, d) \cong C_k(d, d)$ (see [Mar93, Thm 4.3.6]).

2.3. The nilpotent cone for GL_d . For any commutative \mathbb{C} -algebra R , the R -points of \mathfrak{g}_d is the set of endomorphisms of the free R -module R^d . The R -points of the quotient stack $[\mathfrak{g}_d/G_d]$ is the groupoid whose objects are endomorphisms of locally free R -modules of rank d and morphisms are isomorphisms between such pairs.

Let $\mathcal{N}_d \subset \mathfrak{g}_d$ denote the nilpotent cone, the variety parameterizing nilpotent endomorphisms of \mathbb{C}^d . For R a commutative \mathbb{C} -algebra, the set of R -points of \mathcal{N} is the set of nilpotent endomorphisms of R^d and the groupoid of R -points of $[\mathcal{N}_d/G_d]$ consists of nilpotent endomorphisms of locally free R -modules of rank d and isomorphism between them.

We will be interested in the category $P_{G_d}(\mathcal{N}_d; k)$ of equivariant perverse sheaves on the nilpotent cone. The G_d -orbits stratify \mathcal{N} and the orbits are labelled by partitions of d according to the Jordan decomposition.

2.3.1. We now list some basic facts about the Springer and Grothendieck resolutions of \mathcal{N} and \mathfrak{g} . For a more detailed account, see [CG97, Chapter 3].

Recall that if $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, then Chevalley's restriction theorem tells us that $\mathfrak{g}/G \cong \mathfrak{h}/W$. Let $\mathfrak{h}_{\text{reg}} \subset \mathfrak{h}$ be the complement of the root hyperplanes.

When $G = G_d$, $\mathfrak{h}/W = \mathbb{A}^d/\mathfrak{S}_d =: \mathbb{A}^{(d)}$ is naturally isomorphic to the affine space of monic polynomials of degree d . In this case, $\mathfrak{h}_{\text{reg}} = (\mathbb{A}^{(d)})_{rs}$ the open subvariety of monic polynomials with d distinct roots.

Let $\chi : \mathfrak{g} \rightarrow \mathfrak{g}/G$ be the quotient map. In the case of G_d , χ is the map sending an endomorphism to its characteristic polynomial. Note that $\chi^{-1}(0) = \mathcal{N}$.

Recall the existence of Grothendieck's simultaneous resolution $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ of the fibers of the map χ , which fits into the following diagram:

$$\begin{array}{ccccc}
 & \tilde{\mathcal{N}} & \xrightarrow{\quad} & \tilde{\mathfrak{g}} & \\
 \pi_{\mathcal{N}} \swarrow & & & \searrow \pi & \\
 \mathcal{N} & \xrightarrow{\quad} & \mathfrak{g} & & \mathbb{A}^d \\
 & \searrow & \searrow \chi & & \swarrow \\
 & \{0\} & \xrightarrow{\quad} & \mathbb{A}^{(d)} &
 \end{array}$$

The map $\pi_{\mathcal{N}}$ is semi-small and π is small. The restriction of the map χ to $\mathfrak{g}_{rs} = \chi^{-1}(\mathfrak{h}_{\text{reg}})$ is smooth with fibers isomorphic to G/T . The space $\mathfrak{h}_{\text{reg}}$ has fundamental group the braid group \tilde{W} . As the fibers of χ over \mathfrak{h}_{rs}/W are simply connected, the fundamental group of \mathfrak{g}_{rs} is also the braid group. The restriction of π to $\tilde{\mathfrak{g}}_{rs}$ is the W -cover corresponding to the pure subgroup.

2.4. Springer theory. Here we review Juteau's study of modular Springer theory [Jut07]. Before doing so, we remark that while Juteau works with varieties over finite fields, we consider the analogous situation over the complex numbers.

Let $\mathcal{S} = \pi_{\mathcal{N}*} \mathbf{IC}_{\tilde{\mathcal{N}}}$ denote the Springer sheaf and $\mathcal{G} = \pi_* \mathbf{IC}_{\tilde{\mathfrak{g}}}$ the Grothendieck sheaf.

There exists a *Fourier-Sato transform* functor, denoted

$$\mathbb{T}_{\mathfrak{g}} : D_G(\mathfrak{g}; k) \rightarrow D_G(\mathfrak{g}^*; k).$$

It is defined by composing the functor defined in [KS94, §3.7] with the shift $[n^2]$. With this shift, \mathbb{T} is t -exact for the perverse t -structure [KS94, Proposition 10.3.18]. This functor is an equivalence of categories with inverse

$${}^{\vee}\mathbb{T}_{\mathfrak{g}} : D_G(\mathfrak{g}^*; k) \rightarrow D_G(\mathfrak{g}; k).$$

We fix an isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ and will identify them from now on. The following is due to Juteau in the modular case:

Proposition 2.6. *There is a natural isomorphism $\mathbb{T}_{\mathfrak{g}}(\mathcal{S}) \cong \mathcal{G}$.*

Using this, Juteau defines a modular Springer correspondence from irreducible representation of W to \mathbf{IC} -sheaves on \mathcal{N} . The correspondence associates to an irreducible W -representation the image of the functor $\mathbb{T}_{j_{rs!}*}$ applied to the corresponding local system on \mathfrak{g}_{rs} . The \mathbf{IC} -sheaves that occur in the correspondence are precisely those contained in the top (or equivalently in the socle) of the Springer sheaf.

The Grothendieck sheaf \mathcal{G} carries a natural W -action because it is a Goresky-MacPherson extension of a pushforward along a W -cover. Using this action, we can equip the Springer sheaf \mathcal{S} with a W -action in two ways: by the Fourier transform or by restriction to the nilpotent cone.

Proposition 2.7. *The two W -actions on \mathcal{S} - one defined by Fourier transform and the other by restriction to the nilpotent cone - differ by the sign character. In other words, there is an isomorphism of W -sheaves:*

$$\mathbb{T}_{\mathfrak{g}}(\mathcal{G}) \cong i^* \mathcal{G} \otimes \text{sgn}.$$

A version of this proposition for \mathbb{Q}_ℓ -sheaves appeared in [Hot81] and was proven for \mathcal{D} -modules in [Gin83] and [HK84]. A proof in the modular case is to appear in work of Achar, Henderson, and Riche.

2.5. The affine Grassmannian. We first recall the affine Grassmannian for GL_n , which will play a role analogous to \mathcal{N}_d , and then the Beilinson-Drinfeld Grassmannian, which corresponds under the same analogy to \mathfrak{g}_d .

2.5.1. Local version. Let $\mathcal{G}r$ denote the affine Grassmannian over \mathbb{C} for the group $G_n = GL_n$. In other words, $\mathcal{G}r$ is the ind-scheme over \mathbb{C} whose R -points are the set $G_n(\mathcal{K})/G_n(\mathcal{O})$ where \mathcal{K} is the ring of Laurent series $R((t))$ and \mathcal{O} its ring of Laurent polynomials, $R[[t]]$, for R a commutative \mathbb{C} -algebra. The group $G_n(\mathcal{K})$ acts transitively on the set of \mathcal{O} -lattices in $\mathcal{K}^{\oplus n}$, and the stabilizer of the standard lattice is $G_n(\mathcal{O})$. It follows that one can view the R -points of $\mathcal{G}r$ as lattices.

For each $\lambda \in \Lambda^+$, we associate the corresponding $G_n(\mathcal{O})$ -orbit in $\mathcal{G}r$, denoting it by $\mathcal{G}r^\lambda$. Let $\mathbf{d} = (d, 0, \dots, 0) \in \mathbb{Z}^n$. We will be interested in $\overline{\mathcal{G}r^{\mathbf{d}}}$ and $G_n(\mathcal{O})$ -perverse sheaves supported on it. Let \mathcal{O}_d denote the quotient $\mathcal{O}/t^d \mathcal{O}$. As the congruence subgroup $\text{Ker}(G_n(\mathcal{O}) \rightarrow G_n(\mathcal{O}_d)) = 1 + t^d \mathfrak{g}_d(\mathcal{O})$ acts trivially on $\overline{\mathcal{G}r^{\mathbf{d}}}$, it is equivalent to study $P_{G_n(\mathcal{O}_d)}(\overline{\mathcal{G}r^{\mathbf{d}}}; k)$. From the definitions above, one finds that $\overline{\mathcal{G}r^{\mathbf{d}}}$ is a projective variety parameterizing lattices L contained in the standard lattice $L_0 = \mathcal{O}^{\oplus n}$ such that the quotient L_0/L is locally free of rank d .

Considering the $G_n(\mathcal{O})$ -orbits in $\overline{\mathcal{G}r^{\mathbf{d}}}$, we find that, as $n \geq d$, they are labelled by partitions of d .

Let ϖ_1 be the fundamental weight $(1, 0, \dots, 0)$. There is a $G(\mathcal{O})$ -equivariant semi-small resolution $(\mathcal{G}r^{\varpi_1})^{*d} \rightarrow \overline{\mathcal{G}r^{\mathbf{d}}}$. The \mathbb{C} -points of $(\mathcal{G}r^{\varpi_1})^{*d}$ are given by all flags $0 \subset V^1 \subset V^2 \subset \dots \subset V^{d-1} \subset L_0/L$ preserved by the action of \mathcal{O} .

2.5.2. Global version. The d -th Beilinson-Drinfeld Grassmannian [BD, MV07a] of GL_n , $\mathfrak{G}(n, d)$, is an indscheme defined over $\mathbb{A}^{(d)} = \mathbb{A}^d/\mathfrak{S}_d$ whose points are isomorphism classes of triples (x, \mathcal{F}, β) where $x \in \mathbb{A}^{(d)}$, \mathcal{F} is a rank n vector bundle on \mathbb{A}^1 , and β is a trivialization of \mathcal{F} away from $\cup x \subset \mathbb{A}^1$.

Let \mathfrak{K} be the ring of rational functions $R(t)$ and \mathfrak{O} the ring of polynomials $R[t]$. Let \mathcal{L}_0 denote the standard \mathfrak{O} -lattice in $\mathfrak{K}^{\oplus n}$.

Following [Ngô99, MV07b], we will be interested in a particular closed subscheme of the $\mathfrak{G}(n, d)$ which we denote by \mathfrak{G}_d . The R -points of \mathfrak{G}_d are the \mathfrak{O} -lattices $\mathcal{L} \subset \mathcal{L}_0$ such that $\mathcal{L}_0/\mathcal{L}$ is a locally free R -module of rank d . To see that this is a subfunctor of $\mathfrak{G}(n, d)$, note that any such lattice is a locally free \mathfrak{O} -module of rank n , i.e., a vector bundle of rank n on \mathbb{A}^1 .

Equivalently, the points of \mathfrak{S}_d can be expressed as

$$\mathfrak{S}_d(R) = \{g\mathcal{L}_0 \subset \mathcal{L}_0 \mid g \in G_n(\mathfrak{K}) \cap \mathfrak{g}_n(\mathfrak{O}), \deg(\det(g))\}.$$

From this point of view, the natural map to $\mathbb{A}^{(d)}$ is defined by sending a lattice $\mathcal{L} = g\mathcal{L}_0$ to the determinant of g .

Let $G_{n,d}$ be the group scheme over $\mathbb{A}^{(d)}$ whose fiber at a point $P \in \mathbb{A}^{(d)}(R)$ (thought of as a monic polynomial) has R -points $G_{n,d}(P)(R) = GL(\mathfrak{O}/(P))(R)$.

Ngô checks in [Ngô99, 2.1.1], that $G_{n,d} \rightarrow \mathbb{A}^{(d)}$ is smooth with geometrically connected fibers of dimension n^2d . It comes with a natural action $G_{n,d} \times_{\mathbb{A}^{(d)}} \mathfrak{S}_d \rightarrow \mathfrak{S}_d$.

We will consider the stack $[\mathfrak{S}_d/G_{n,d}]$. Its R -points form the groupoid whose objects are pairs $(L \subset F)$ where F is a locally free \mathfrak{O} -module of rank n with L a submodule, such that F/L is locally free over R of rank d . Again, $[\mathcal{G}^{\mathbf{d}}/G_n(\mathcal{O}_d)]$ is then simply the subfunctor of such pairs where $P = x^d$.

Analogous to the Grothendieck resolution, there is a global resolution, which we denote by $\tilde{\mathfrak{S}}_d$. It is the scheme whose R -points are full flags of \mathfrak{O} -lattices.

Observe that the stalks of \mathfrak{S}_d , $G_{n,d}$, and $\tilde{\mathfrak{S}}_d$ over $0 \in \mathbb{A}^{(d)}$ are respectively $\overline{\mathcal{G}^{\mathbf{d}}}$, $G_n(\mathcal{O}_d)$, and $(\mathcal{G}^{\varpi_1})^{*d}$. We thus see that the spaces described fit into a diagram analogous to that of Springer theory.

$$\begin{array}{ccccc} & & (\mathcal{G}^{\mathbf{d}})^{*d} & \xrightarrow{\quad} & \tilde{\mathfrak{S}}_d \\ & \swarrow \pi_{\mathcal{G}^{\mathbf{d}}} & & \searrow \pi_{\mathfrak{S}} & \downarrow \tilde{f} \\ \overline{\mathcal{G}^{\mathbf{d}}} & \xrightarrow{\quad} & \mathfrak{S}_d & \xrightarrow{\quad f \quad} & \mathbb{A}^d \\ & \searrow & & \swarrow & \\ & & \{0\} & \xrightarrow{\quad} & \mathbb{A}^{(d)} \end{array}$$

Similarly to above, $\pi_{\mathcal{G}^{\mathbf{d}}}$ is semi-small and $\pi_{\mathfrak{S}}$ is small. Let $\mathfrak{S}_{\text{rs}} = f^{-1}((\mathbb{A}^{(d)})_{\text{rs}})$, $\tilde{\mathfrak{S}}_{\text{rs}} = \pi_{\mathfrak{S}}^{-1}(\mathfrak{S}_{\text{rs}})$, and $j_{\mathfrak{S}_{\text{rs}}} : \mathfrak{S}_{\text{rs}} \rightarrow \mathfrak{S}$ be the inclusion. The restriction of $\pi_{\mathfrak{S}}$, $\pi_{\mathfrak{S},\text{rs}} : \tilde{\mathfrak{S}}_{\text{rs}} \rightarrow \mathfrak{S}_{\text{rs}}$, is a Galois \mathfrak{S}_d -cover. It follows that $\pi_{\mathfrak{S}} \mathbf{IC}_{\tilde{\mathfrak{S}}} \cong (j_{\mathfrak{S}_{\text{rs}}})_! (\pi_{\mathfrak{S}_{\text{rs}}})^* \mathbf{IC}_{\tilde{\mathfrak{S}}_{\text{rs}}}$.

2.6. Geometric Satake. Mirkovic and Vilonen [MV07a] prove, following work in characteristic zero of Lusztig [Lus83] and then Ginzburg [Gin95], that for a split reductive group G defined over a commutative ring k : the category $P_{G^\vee(\mathcal{O})}(\mathcal{G}; k)$ of equivariant perverse sheaves with coefficients in k on the affine Grassmannian for the Langlands dual group G^\vee/\mathbb{C} , together with its convolution structure, is tensor equivalent to the category of representation of G over k .

In this paper, we only consider the case $G = G^\vee = GL_n$. We use the following immediate corollaries of the results of [MV07a].

Corollary 2.8. *There is an equivalence of categories*

$$P_{G_n(\mathcal{O})}(\overline{\mathcal{G}^{\mathbf{d}}}; k) \cong \mathcal{C}_k(n, d).$$

Under this equivalence, the GL_n -representation $E^{\otimes d}$ (here E is the standard n -dimensional representation) corresponds to the restriction of $\pi_{\mathfrak{S}} \mathbf{IC}_{\tilde{\mathfrak{S}}}$ to \mathcal{G} , or equivalently to the pushforward $\pi_{\mathcal{G}^{\mathbf{d}}} \mathbf{IC}_{(\mathcal{G}^{\mathbf{d}})^{*d}}$.*

The symmetric group \mathfrak{S}_d acts on

$$\pi_{\mathcal{G}r*} \mathbf{IC}_{(\mathcal{G}r^{\varpi_1})^{*d}} \cong [(j_{\mathfrak{G},rs})!_*(\pi_{\mathfrak{G},rs})! \mathbf{IC}_{\tilde{\mathfrak{G}}_{rs}}]||_{\mathcal{G}r}$$

by the deck transformations of $\pi_{\mathfrak{G},rs}$ and the functoriality of $!*$ -extensions. This action corresponds under the geometric Satake equivalence to the permutation action of \mathfrak{S}_d on $E^{\otimes d}$.

3. A PROJECTION MAP AND LUSZTIG'S SECTION

In this section, we relate the spaces \mathcal{N}_d and $\overline{\mathcal{G}r^d}$ and their quotient stacks. We do so using the functor of points descriptions given in the previous sections.

Lemma 3.1. *There exist natural morphisms:*

$$(3.1) \quad \begin{array}{ccc} \overline{\mathcal{G}r^d} & \xleftarrow{\overline{\phi}} & \mathcal{N}_d \\ \downarrow & \searrow \tilde{\psi} & \downarrow \\ [\overline{\mathcal{G}r^d}/G_n(\mathcal{O}_d)] & \xrightarrow[\phi]{\psi} & [\mathcal{N}_d/G_d] \end{array} \quad \begin{array}{ccc} \mathfrak{G}_d & \xleftarrow{\overline{\Phi}} & \mathfrak{g}_d \\ \downarrow & \searrow \tilde{\Psi} & \downarrow \\ [\mathfrak{G}_d/G_{n,d}] & \xrightarrow[\Phi]{\Psi} & [\mathfrak{g}_d/G_d] \end{array}$$

such that all of the solid morphisms form a commutative diagram, while the dotted morphisms satisfy $\psi \circ \phi = Id_{[\mathcal{N}_d/G_d]}$ and $\Psi \circ \Phi = Id_{[\mathfrak{g}_d/G_d]}$.

Proof. Recall diagrams 2.3.1 and 2.5.2. In the lemma above, the morphisms denoted by lower-case letters will be defined as the restriction of the upper-case morphisms by the closed embeddings $\mathcal{N}_d \rightarrow \mathfrak{g}_d$ and $\overline{\mathcal{G}r^d} \rightarrow \mathfrak{G}_d$. We thus need only describe the global (or upper-case) morphisms and check that they satisfy the relations described.

The maps $\tilde{\Psi}$ and $\tilde{\psi}$. The map $\tilde{\Psi}$ is the forgetful map which associates to an R -point of \mathfrak{G}_d , $\mathcal{L} \in \mathfrak{G}_d(R)$, the rank d locally free R -module $\mathcal{L}_0/\mathcal{L}$ together with the endomorphism given by the action of $t \in \mathfrak{D}$. Similarly, $\tilde{\psi}$ associates to a point $(P, L \subset F)$ of $[\mathfrak{G}_d/G_{n,d}]$, the locally free R -module F/L together with the endomorphism defined by multiplication by t .

The maps $\overline{\Phi}$ and $\overline{\phi}$. To any R -point of $[\mathfrak{g}_d/G_d]$, an endomorphism a of a locally free rank d R -module E , let $\Phi(a)$ be the pair $(L \subset F)$, where $F = E[t]$ and $L = (a - t)E[t]$. To see that this is indeed a point of \mathfrak{G}_d , note that $a - t \in GL(E(t)) \cap \mathfrak{gl}(E[t])$ and that its determinant is the characteristic polynomial of a and therefore of degree d as desired.

Remark 3.2. The local version $\overline{\phi}$ was first observed by Lusztig [Lus81]. As far as I am aware, a description of the global map $\overline{\Phi}$ first appeared in [MV07b].

Remark 3.3. Lusztig [Lus81] observes that the map $\overline{\phi}$ is an open embedding when $n = d$ and the same is true of $\overline{\Phi}$. Moreover, the image of the imbedding intersects every orbit and provides a natural bijection between the orbits of $\overline{\mathcal{G}r^d}$ and those of \mathcal{N}_d .

It remains to exhibit a natural equivalence $id \xrightarrow{\sim} \Psi \circ \Phi$.

We claim that such an equivalence can be defined as follows. For any (a, E) , consider the map of R -modules given by

$$E \hookrightarrow E[t] \rightarrow E[t]/(a - t)E[t].$$

Note that the map is injective as $E \cap (a - t)E[t] = 0$. To prove surjectivity, one can use induction on the degree to show that any polynomial $p(t) \in E[t]$ is in $E + (a - t)E[t]$. To finish the proof, we observe that this isomorphism intertwines the action of a on W with the action of t on $E[t]/(a - t)E[t]$. \square

4. EQUIVALENCE OF CATEGORIES

Fix $n \geq d$.

4.1. In this section we use the Lemma 3.1 from the previous section to prove certain equivalences of categories.

Theorem 4.1. *The maps $\phi : \mathcal{N}_d \rightarrow \overline{\mathcal{G}r^{\mathbf{d}}}$ and $\Phi : \mathfrak{g}_d \rightarrow \mathfrak{G}_d$ give equivalence of categories*

$$\phi^\bullet := \phi^*[d^2 - nd] : P_{G_n(\mathcal{O})}(\overline{\mathcal{G}r^{\mathbf{d}}}) \xrightarrow{\sim} P_{G_d}(\mathcal{N}_d),$$

$$\Phi^\bullet := \Phi^*[d^2 - nd] : P_{G_{n,d}}(\mathfrak{G}_d) \xrightarrow{\sim} P_{G_d}(\mathfrak{g}_d).$$

Proof. By [BBD82, 4.2.5], the (shifted) pull-back along a smooth morphism $f : X \rightarrow Y$ with geometrically connected fibers is a fully faithful embedding of the category of perverse sheaves on Y to that on X .

We now check that Φ enjoys these properties (and therefore ϕ does as well by base change). From this it will follow that Φ^\bullet and ϕ^\bullet are fully faithful.

First consider the case $n = d$. Recall that $\overline{\phi}, \overline{\Phi}$ are an open morphisms (Remark 3.3). From this, we can deduce that ϕ and Φ are smooth. As the intersection of \mathfrak{g} is open and connected in every orbit of \mathfrak{G} , we conclude that the fibers are geometrically connected.

In the general case $n > d$, we can factor $\tilde{\Phi}_d^n$ as $\tilde{\Phi}_d^d$ and the map $a_{d,n} : [\mathfrak{G}_d/G_{d,d}] \rightarrow [\mathfrak{G}_n/G_{n,d}]$. In [Ngô99, Lemme 2.2.1], it is shown that $a_{d,n}$ is smooth with connected fibers.

By Lemma 3.1, the compositions $\phi^* \circ \psi^*$ and $\Phi^* \circ \Psi^*$ are the identity functors on $P_{G_d}(\mathcal{N}_d)$ and $P_{G_{n,d}}(\mathfrak{G}_d)$ respectively. We conclude that ϕ^\bullet and Φ^\bullet are also essentially surjective, which completes the proof. \square

4.2. We conclude this section with an observation that we will not use in what follows, but which puts in perspective the relationship between the various equivariant derived categories.

Corollary 4.2. *The pullback functor on equivariant derived categories*

$$\psi^* : D_{G_d}(\mathcal{N}_d) \rightarrow D_{G_n(\mathcal{O})}(\overline{\mathcal{G}r^{\mathbf{d}}})$$

(resp. $\Psi^* : D_{G_d}(\mathfrak{g}_d) \rightarrow D_{G_{n,d}}(\mathfrak{G}_d)$) splits the functor

$$\phi^* : D_{G_n(\mathcal{O})}(\overline{\mathcal{G}r^{\mathbf{d}}}) \rightarrow D_{G_d}(\mathcal{N}_d)$$

(resp. $\Phi^* : D_{G_{n,d}}(\mathfrak{G}_d) \rightarrow D_{G_d}(\mathfrak{g}_d)$), in the following sense:

For any two objects, $A, B \in D_G(\mathcal{N})$, ψ^* induces an injection of graded vector spaces

$$\mathrm{Ext}_{D_G(\mathcal{N})}^*(A, B) \hookrightarrow \mathrm{Ext}_{D_{G(\mathcal{O})}(\overline{\mathcal{G}r^{\mathbf{d}}})}^*(\psi^* A, \psi^* B),$$

which naturally splits the projection map given by ϕ^* .

Remark 4.3. It is worth noting that as functors between equivariant derived categories, the pullback functors do not induce equivalences, and in fact the various categories are not equivalent. To see this, consider the Ext-groups between the standard and costandard sheaf on the stratum associated to a partition λ . The Ext groups in the equivariant derived categories will agree with the equivariant cohomology of the stratum, which in turn agrees with the group cohomology (of the reductive part) of the stabilizer of a point. Unless $n = d$ and λ is the trivial partition, these will not agree.

5. GENERALIZED SCHUR-WEYL THEOREM

In this section we give a new, geometric proof of Carter and Lusztig's Schur-Weyl duality with general coefficients [CL74, Thm. 3.1].

Theorem 5.1. *Let GL_n^k denote GL_n over a ring k and E be the standard n -dimensional representation. For any $d \leq n$, the action of \mathfrak{S}_d on $E^{\otimes d}$ induces an isomorphism*

$$k[S_d] \rightarrow \text{End}_{GL_n^k}(E^{\otimes d}).$$

Proof. Recall the following consequences of the geometric Satake equivalence summarized in Corollary 2.8. There is natural isomorphism

$$\text{End}_{GL_n^k}(E^{\otimes d}) \cong \text{End}_{P_{G(\mathcal{O})}(\mathcal{G}_r)}(\pi_{\mathcal{G}_r*} \mathbf{IC}_{(\mathcal{G}_r^1)^{*d}}).$$

The perverse sheaf $\pi_{\mathcal{G}_r*} \mathbf{IC}_{(\mathcal{G}_r^1)^{*d}} \cong [(j_{\mathfrak{S}_{rs}})_! (\pi_{\mathfrak{S}_{rs}}^* \mathbf{IC}_{\tilde{\mathfrak{S}}_{rs}})]|_{\mathcal{G}_r}$ carries an action of \mathfrak{S}_d induced by the action of the deck-transformations of $\pi_{\mathfrak{S}_{rs}}$, and this action agrees under the isomorphism above with the permutation action of \mathfrak{S}_d on $E^{\otimes d}$.

We wish to translate this action from the affine Grassmannian to the nilpotent cone. A simple generalization of the maps Φ and $\bar{\phi}$ completes the following commutative cube:

$$(5.1) \quad \begin{array}{ccccc} & \tilde{\mathcal{N}}_d & \xrightarrow{\quad} & \tilde{\mathfrak{g}}_d & \\ & \swarrow & & \searrow & \\ \mathcal{N}_d & \xrightarrow{\quad} & \mathfrak{g}_d & \xrightarrow{(\mathcal{G}_r^{\varpi_1})^{*d}} & \tilde{\mathfrak{S}}_d \\ & \searrow & \swarrow & \searrow & \\ & \overline{\mathcal{G}_r^d} & \xrightarrow{\quad} & \mathfrak{S}_d & \end{array}$$

Here the left and right faces are both pull-back squares, all the maps from upper-right to lower-left are proper, and the maps from upper-left to lower-right are inclusions.

By Theorem 4.1, the functor ϕ^\bullet induces an isomorphism

$$\text{End}_{P_{G(\mathcal{O})}(\mathcal{G}_r)}(\pi_{\mathcal{G}_r*} \mathbf{IC}_{(\mathcal{G}_r^1)^{*d}}) \cong \text{End}_{P_{G(\mathcal{O})}(\mathcal{N})}(\mathcal{S}).$$

The \mathfrak{S}_d -action on $\pi_{\mathcal{G}_r*} \mathbf{IC}_{(\mathcal{G}_r^1)^{*d}}$ corresponds to the analogously-defined action on $\mathcal{S} \cong i^*(j_{rs})_! \pi_{rs*} \mathbf{IC}_{\tilde{\mathfrak{S}}_{rs}}$.

On the other hand, Juteau [Jut07] observes that, as in characteristic 0, the Fourier transform exchanges the Springer and Grothendieck sheaves and so

$$\text{End}(\mathcal{S}) \cong \text{End}(\mathcal{G}) \cong \text{End}(\pi_{rs!} \mathbf{IC}_{\tilde{\mathfrak{S}}_{rs}}) \cong k[\mathfrak{S}_d].$$

Proposition 2.7 says that the resulting action of \mathfrak{S}_d by Fourier transform differs from the one arising by restriction (as in the previous paragraph) by a sign character. But as the first induces an isomorphism, the second does as well. \square

6. GEOMETRIC SCHUR FUNCTOR

The previous section allows us to identify the functor $\mathrm{Hom}(\mathcal{S}, -)$ with the Schur functor. In other words, we have a commutative diagram of functors:

$$(6.1) \quad \begin{array}{ccc} P_{G_d}(\mathcal{N}_d) & \xrightarrow{\mathrm{Hom}(\mathcal{S}, -)} & \mathrm{Mod} - \mathrm{End}(\mathcal{S}) \\ \downarrow \cong & & \uparrow \cong \\ C_k(n, d) & \xrightarrow{\mathrm{Hom}(E^{\otimes d}, -)} & \mathrm{Mod} - k[\mathfrak{S}_d], \end{array}$$

where the action of \mathfrak{S}_d on \mathcal{S} is through the action by deck transformations on \mathcal{G} and the identification of \mathcal{S} with $i^*\mathcal{G}$.

Motivated by this connection, we will shift our focus to a general connected complex reductive group. For a general complex reductive group G , by analogy, we can define a *Schur functor* from category of G -equivariant perverse sheaves on the nilpotent cone in \mathfrak{g} to representations of the Weyl group,

$$\mathrm{Hom}(\mathcal{S}, -) : P_G(\mathcal{N}_G; k) \rightarrow \mathrm{Mod} - \mathrm{End}(\mathcal{S}).$$

In this section, we will introduce another functor that is closely related to $\mathrm{Hom}(\mathcal{S}, -)$. We propose that this new functor, which also exists for any complex reductive group, should be viewed as a geometric Schur functor.

For any connected reductive group G , consider the functor:

$$\mathcal{F} := j_{rs}^* \circ \mathbb{T}_{\mathfrak{g}} \circ (i_{\mathcal{N}})_* : P_G(\mathcal{N}) \rightarrow P_G(\mathfrak{g}_{rs}).$$

Note that it is exact because each component is exact. To ease notation, we will let $\mathbb{T} = \mathbb{T}_{\mathfrak{g}}$, $j = j_{rs}$ and $i = i_{\mathcal{N}}$. Let $\mathrm{Loc}_W(\mathfrak{g}_{rs}) \subset P_G(\mathfrak{g}_{rs})$ denote the full subcategory of local systems with monodromy factoring through the Weyl group.

In the proof of the following lemma we will use parabolic restriction and induction functors. For their definition and some basic properties in this context, see for example [AM12].

Lemma 6.1. *The functor \mathcal{F} factors through the inclusion $\mathrm{Loc}_W(\mathfrak{g}_{rs}) \subset P_G(\mathfrak{g}_{rs})$.*

Proof. We first show that the statement is true for all simple objects in $P_G(\mathcal{N}; k)$. For each nilpotent orbit \mathcal{O} and irreducible local system $\mathcal{L} \in P_G(\mathcal{O}; k)$, consider the **IC**-sheaf $A = \mathbf{IC}(\mathcal{O}, \mathcal{L})$. Let T be a maximal torus for G . Note that $\mathrm{res}_T^G A \neq 0$ if and only if A is in Juteau's modular Springer correspondence and hence $\mathbb{T}A \cong \mathbf{IC}(\mathfrak{g}_{rs}, \mathcal{L}_V)$, where $\mathcal{L}_V \in \mathrm{Loc}_W(\mathfrak{g}_{rs})$ and corresponds to some irreducible $k[W]$ -representation V .

Suppose instead that $\mathrm{res}_T^G A = 0$. Then there exist some Levi subgroup L such that $A_0 = \mathrm{res}_L^G A$ is cuspidal. It follows that $\mathbb{T}A_0$ is also cuspidal and by [Mir04, Lemma 4.4]¹, every cuspidal sheaf of \mathfrak{l} is supported on $\mathcal{N}_L \times Z(\mathfrak{l})$. By adjunction, there exists a non-zero map $A \rightarrow \mathrm{ind}_L^G A_0$. As A is simple,

$$\overline{\mathrm{supp} \mathbb{T}A} \subset \overline{\mathrm{ind}_L^G \mathbb{T}A_0} \subset {}^G(\mathcal{N}_L \times Z(\mathfrak{l})).$$

¹While Mirković works with \mathcal{D} -modules, the same argument works in the constructible context with arbitrary coefficients.

Finally, ${}^G(\mathcal{N}_L \times Z(\mathfrak{l})) \cap \mathfrak{g}_{rs} = \emptyset$ as L is not a maximal torus. Thus $j^* \mathbb{T}A = 0$.

From this we conclude that the functor \mathcal{F} at least factors through the category of G -equivariant local systems on \mathfrak{g}_{rs} . It remains to check that for a general object $X \in P_G(\mathcal{N})$, $\mathcal{F}X$ has monodromy that factors through W .

Let $Q \subset X$ be minimal such that $\mathcal{F}(X/Q) = 0$ (Such an object exists because the category of perverse sheaves is Artinian). By the exactness of \mathcal{F} , $\mathcal{F}(Q) \cong \mathcal{F}(X)$. By the minimality of Q , the top of Q is a direct sum of \mathbf{IC} -sheaves in Juteau's modular Springer correspondence. Recall that the Springer sheaf is a direct sum of the projective covers of such \mathbf{IC} -sheaves. Thus for some $m \geq 0$, there exists a surjection $\mathcal{S}^{\oplus m} \rightarrow Q$. Using again the exactness of \mathcal{F} , we conclude that $\mathcal{F}Q$ is a quotient of m copies of $j^* \mathcal{G}$ and thus has monodromy that factors through the Weyl group. \square

In order to compare \mathcal{F} with $\text{Hom}(\mathcal{S}, -)$, we consider the functor

$$\rho : \text{Mod} - \text{End}(j^* \mathcal{G}) \rightarrow \text{Loc}_W(\mathfrak{g}_{rs})$$

defined by the tensor product $(-) \otimes_{\text{End}(j^* \mathcal{G})} j^* \mathcal{G}$ and its inverse

$$\text{Hom}(j^* \mathcal{G}, -) : \text{Loc}_W(\mathfrak{g}_{rs}) \rightarrow \text{Mod} - \text{End}(j^* \mathcal{G}).$$

Theorem 6.2. *There is a natural equivalence of functors:*

$$(6.2) \quad \begin{array}{ccc} P_G(\mathcal{N}) & \xrightarrow{\mathcal{F}} & \text{Loc}_W(\mathfrak{g}_{rs}) \\ & \searrow \text{Hom}(\mathcal{S}, -) & \nearrow \sim \\ & \text{Mod} - \text{End}(\mathcal{S}) \cong \text{Mod} - \text{End}(j^* \mathcal{G}) & \end{array}$$

Proof. We proceed by constructing a sequence of equivalences. Fourier transform induces an equivalence of functors:

$$\rho(\text{Hom}(\mathcal{S}, -)) \cong \rho(\text{Hom}(\mathcal{G}, \mathbb{T}i_*(-))).$$

For any $X \in P_G(\mathcal{N})$, the Fourier transform $\mathbb{T}X$ is perverse and so we obtain an exact sequence:

$$0 \rightarrow K \rightarrow \mathbb{T}X \rightarrow {}^p j_* j^* \mathbb{T}X \rightarrow C \rightarrow 0$$

where K and C have support in $\mathfrak{g} - \mathfrak{g}_{rs}$. Applying the exact functor $\text{Hom}(\mathcal{G}, -)$ gives an isomorphism $\text{Hom}(\mathcal{G}, \mathbb{T}X) \cong \text{Hom}(\mathcal{G}, {}^p j_* j^* \mathbb{T}X)$. Using the adjunction between j^* and ${}^p j_*$ we obtain an equivalence:

$$\rho(\text{Hom}(\mathcal{G}, \mathbb{T}i_*(-))) \cong \rho(\text{Hom}(j^* \mathcal{G}, j^* \mathbb{T}i_*(-))).$$

The composition $\rho \circ \text{Hom}(j^* \mathcal{G}, -)$ restricted to $\text{Loc}_W(\mathfrak{g}_{rs})$ is isomorphic to the identity functor. From Lemma 6.1, the functor $j^* \mathbb{T}i_*$ takes values in $\text{Loc}_W(\mathfrak{g}_{rs})$ and so we may conclude:

$$\rho(\text{Hom}(j^* \mathcal{G}, j^* \mathbb{T}i_*(-))) \cong j^* \mathbb{T}i_*(-) = \mathcal{F}(-).$$

\square

7. GEOMETRIC ADJOINT FUNCTORS

In the previous section we introduced the notion of a geometric Schur functor for an arbitrary connected complex reductive group. We close with a simple observation in this general context.

By the usual adjointness for closed and open embeddings, the geometric Schur functor has the following left and right adjoints which we consider restricted to $\text{Loc}_W(\mathfrak{g}_{rs})$:

$$\mathcal{G}_R = {}^p i^! \circ {}^{\vee} \mathbb{T}_{\mathfrak{g}} \circ {}^p j_* : \text{Loc}_W(\mathfrak{g}_{rs}) \rightarrow P_G(\mathcal{N}),$$

$$\mathcal{G}_L = {}^p i^* \circ {}^{\vee} \mathbb{T}_{\mathfrak{g}} \circ {}^p j_! : \text{Loc}_W(\mathfrak{g}_{rs}) \rightarrow P_G(\mathcal{N}).$$

Lemma 7.1. *The functor ${}^{\vee} \mathbb{T}_{j_!} : \text{Loc}_W(\mathfrak{g}_{rs}) \rightarrow P_{G_d}(\mathfrak{g})$ factors through the subcategory $P_{G_d}(\mathcal{N}) \subset P_{G_d}(\mathfrak{g})$.*

In other words, the perverse extension of a representation of the Weyl group is anti-orbital.

Proof. It suffices to check on a projective generator of $\text{Loc}_W(\mathfrak{g}_{rs})$. The local system $j^* \mathcal{G}$ is such a projective generator and $j_{!*} j^* \mathcal{G} \cong \mathcal{G}$. Thus ${}^{\vee} \mathbb{T}_{j_!} j^* \mathcal{G} \cong i_* \mathcal{S}$, which has support on \mathcal{N} . \square

Proposition 7.2. *The adjoint functors $\mathcal{G}_L, \mathcal{G}_R$ are both right inverses to \mathcal{F} .*

Proof. We provide the proof for \mathcal{G}_R , the proof for \mathcal{G}_L is completely analogous.

We seek a natural equivalence $j^* \mathbb{T} i_* {}^p i^! \mathbb{T}^p j_* \cong \text{Id}$. Consider an element $\mathcal{L} \in \text{Loc}_W(\mathfrak{g}_{rs})$. One has the canonical inclusion $j_{!*} \mathcal{L} \hookrightarrow {}^p j_* \mathcal{L}$. The Fourier transform ${}^{\vee} \mathbb{T}$ is an equivalence of categories, so ${}^{\vee} \mathbb{T}_{j_!} \mathcal{L} \hookrightarrow {}^{\vee} \mathbb{T}^p j_! \mathcal{L}$. By lemma 7.1, the ${}^{\vee} \mathbb{T}_{j_!} \mathcal{L}$ is supported on the nilpotent cone \mathcal{N} . On the other hand, for any $\mathcal{F} \in P(\mathfrak{g})$, the perverse sheaf $i_* {}^p i^! \mathcal{F}$ injects into \mathcal{F} in such a way that any subsheaf \mathcal{A} of \mathcal{F} supported on \mathcal{N} factors through it.

In particular, letting $\mathcal{F} = \mathbb{T}({}^p j_* \mathcal{L})$ and $\mathcal{A} = \mathbb{T}(j_{!*} \mathcal{L})$ we obtain a commutative triangle:

$$(7.1) \quad \begin{array}{ccc} {}^{\vee} \mathbb{T}_{j_!} \mathcal{L} & \xrightarrow{\quad} & {}^{\vee} \mathbb{T}^p j_* \mathcal{L} \\ & \searrow \quad \swarrow & \\ & i_* {}^p i^! \mathbb{T}^p j_* \mathcal{L} & \end{array}$$

We now apply the exact functor $j^* \circ \mathbb{T}$ to obtain:

$$(7.2) \quad \begin{array}{ccc} \mathcal{L} & \xrightarrow{\quad \sim \quad} & \mathcal{L} \\ & \searrow \quad \swarrow & \\ & j^* \mathbb{T} i_* {}^p i^! \mathbb{T}^p j_* \mathcal{L} & \end{array}$$

We conclude that the two bottom maps are also isomorphisms and thus $j^* \circ \mathbb{T}$ applied to the adjunction morphism gives the natural equivalence we sought to prove. \square

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